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Note

Note on the Calculation of Eigenvalues for the Stationary Perturbation of Poiseuille Flow

This paper is an additional note to Bramley and Dennis (*J. Comput. Phys.* **47** (1982), 179) which calculated the eigenvalues for the stationary perturbation of plane Poiseuille flow. Both the velocity/pressure and velocity/voticity formulation of the equations of motion are used instead of the streamfunction formulation. It is found that using the velocity/vorticity formulation more accurate results are obtained than in Bramley and Dennis.

1. Introduction

Bramley and Dennis [1] calculated the real and complex eigenvalues of the stationary perturbation of Poiseuille flow by solving the differential eigenvalue problem

$$\phi^{iv} + 2\alpha^2\phi'' + \alpha^4\phi + \alpha R[\tfrac{3}{2}(1 - y^2)(\phi'' + \alpha^2\phi) + 3\phi] = 0 \tag{1.1}$$

subject to boundary conditions

$$\phi(\pm 1) = \phi'(\pm 1) = 0, \tag{1.2}$$

where R is the Reynolds number, α the eigenvalue and ϕ the eigenfunction. A spectral method using Chebyshev polynomials was used and there was some difficulty in obtaining the results accurate enough for Reynolds numbers 500 to 2000, particularly for complex eigenvalues. To improve matters Bramley and Dennis [2] used initial value value methods to obtain the complex eigenvalues which could not be obtained using the spectral method in [1]. The differential eigenvalue problem (1.1) can be formulated in either velocity/vorticity form or velocity/pressure form instead of the streamfunction formulation as given above. This paper presents the results of [1, 2] using the governing equations in velocity/vorticity and velocity/pressure forms.

The reader is referred to [1] for complete details of the problem. No new results are obtained but it is shown that the results can be more easely obtained by using the velocity/vorticity formulation of equations instead of the streamfunction formulation.

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2. Velocity/Vorticity Formulation

The two-dimensional incompressible viscous flow in a channel obeys the equations

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.1}$$

$$\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}, \tag{2.2}$$

and

$$\frac{\partial^2 \zeta}{\partial x^2}+\frac{\partial^2 \zeta}{\partial y^2}=R\left(u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}\right), \tag{2.3}$$

where x is the (dimensionless) streamwise coordinate, y is the (dimensionless) transverse coordinate and R is the Reynolds number, with a half the channel width Ua the volumetric flow rate of the Poiseuille flow over half the channel width. ζ is the vorticity and u and v are the velocity of the fluid in the x and y directions, respectively. The no-slip boundary conditions are

$$u(\pm 1)=0 \qquad \text{and} \qquad v(\pm 1)=0. \tag{2.4}$$

We now look for the perturbation of the Poiseuille flow where

$$u=\tfrac{3}{2}(1-y^2)+\varepsilon U(y)e^{-\alpha x}, \tag{2.5}$$

$$v=\varepsilon V(y)e^{-\alpha x}, \tag{2.6}$$

$$\zeta=3y+\varepsilon Z(y)^{e-\alpha x}, \tag{2.7}$$

and ε is small. Substituting into Eqs. (2.1)–(2.3) and neglecting squares of ε leads to

$$V'-\alpha U=0, \tag{2.8}$$

$$U'+\alpha V+Z=0, \tag{2.9}$$

$$Z''+\alpha^2 Z+R[\tfrac{3}{2}\alpha Z(1-y^2)-3V]=0, \tag{2.10}$$

with boundary conditions $V(\pm 1)=U(\pm 1)=0$. We now use the Chebyshev spectral method as described in Orszag [3] and Bramley and Dennis [1] to obtain the eigenvalues α. Equations (2.8)–(2.10) uncouple and can be solved over half the channel width with either U even and V, Z odd or U odd and V, Z even. Equations (2.8)–(2.10) are transformed into equations in which the eigenvalue α only occurs in a linear fashion with eigenfunctions U, V, Z, and αZ. The coefficients of the Chebyshev polynomials can now be written in the form

$$(A-\alpha B)\mathbf{b}=0, \tag{2.11}$$

where the vector **b** contains the Chebyshev coefficients of U, V, Z, and αZ. The form of the matrix A is given by Fig. 1 in the case U odd and V, Z even. The two rows of 1's are from the boundary conditions and by simple column operations can be eliminated. The matrix B becomes the unit matrix and we solve to find the eigenvalues of A using a QR algorithm. The case where U is even and V, Z are odd is treated in a similar manner.

The results for the real eigenvalues are the same as those given in Bramley and Dennis [1] except that the eigenvalues for $R = 2000$ are accurate to another decimal place. It is now possible using a spectral method to calculate all of the complex eigenvalues given by Bramley and Dennis [2].

Using 70 terms in either the odd or even Chebyshev expansion one of the three complex eigenvalues was not accurate for $R = 1000$ and two of the three complex eigenvalues were not accurate for $R = 2000$. These eigenvalues could have been calculated to the required accuracy by increasing the number of terms in the

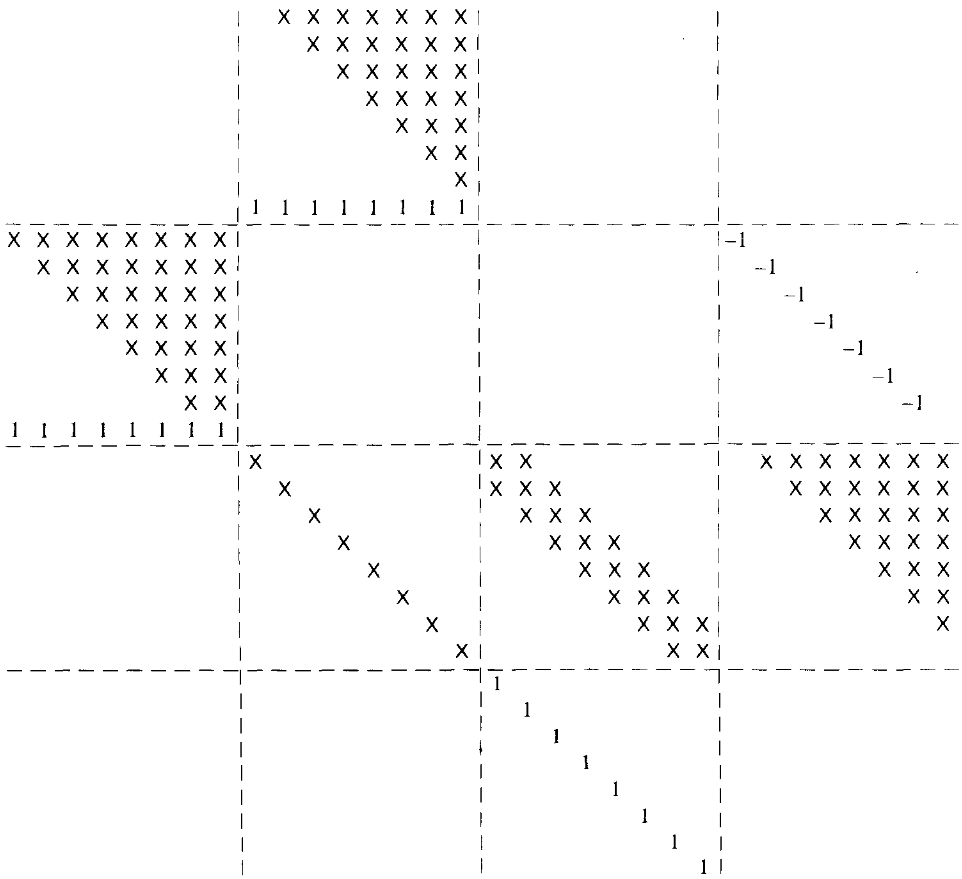

FIG. 1. Structure of matrix A before column operations are performed.

Chebyshev expansion but this was not attempted due to the prohibitive amount of computer time required. In the formulation used in [1] the fourth derivative seems to have caused the problem of the results not getting more accurate as the number of Chebyshev terms was increased. The Chebyshev expansion of the fourth derivative has a term $O(m^7)$, where m is the number of terms in the expansion and this term becomes very large as m is increased. The above formulation splits the fourth derivative into a second derivative and two first derivatives. A second derivative has a term $O(m^3)$ and a first derivative a term $O(m)$. This being so the matrix A is now better conditioned. In the above formulation we still only require the matrix A to be $4m$ square as in the formulation of (1.1). Even though in the streamfunction formulation there is a single equation, due to the term α^4 we needed to introduce eigenfunctions $\alpha\phi$, $\alpha^2\phi$, and $\alpha^3\phi$ in addition to ϕ to obtain an algebraic eigenvalue problem of the form (2.11). In this note the α^2 in (2.10) means that we need to introduce αZ to add to U, V, and Z already defined. If we use the same number of Chebyshev terms for each method the size of A for each method will be the same. Making B into the unit matrix by column operations saves computer time.

3. Velocity/Pressure Formulation

Using the same notation as in Section 2 the governing equations are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{3.1}$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=R\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial p}{\partial x}\right), \tag{3.2}$$

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=R\left(u\frac{\partial v}{\partial x}+u\frac{\partial v}{\partial y}+\frac{\partial p}{\partial y}\right), \tag{3.3}$$

subject to boundary conditions

$$u(\pm 1)=v(\pm 1)=0.$$

We look for a stationary perturbation like

$$u=\tfrac{3}{2}(1-y^2)+\varepsilon U(y)e^{-\alpha x}, \tag{3.4}$$

$$v=\varepsilon V(y)e^{-\alpha x}, \tag{3.5}$$

$$p=p_0-\frac{3x}{R}+\varepsilon P(y)e^{-\alpha x}, \tag{3.6}$$

where ε is small. Substituting (3.4)–(3.6) into (3.1)–(3.3) and nglecting ε^2 we get

$$V' = \alpha U, \tag{3.7}$$

$$U'' + \alpha^2 U + R[P\alpha + 3yV + \tfrac{3}{2}\alpha U(1 - y^2)] = 0, \tag{3.8}$$

$$V'' + \alpha^2 V + R[\alpha V \tfrac{3}{2}(1 - y^2) - P'] = 0, \tag{3.9}$$

with boundary condition $V(\pm 1) = U(\pm 1) = 0$. The α^2 in Eq. (3.8) is eliminated by substituting $\alpha U = V'$ from (3.7).

In order to have α in a linear manner we define an additional variable αV and we now replace U, V, P, and αV by their Chebyshev coefficients and obtain a linear set of algebraic equations

$$(A - \alpha B)\mathbf{b} = 0, \tag{3.10}$$

for each of the cases U, P even, V odd and U, P odd, V even.

The method now follows that described in Section 2 except that it would be more difficult to transform B into a unit matrix by row and column operations. We therefore solve Eq. (3.10) by a QZ algorithm. The method of Section 2 uses about 60% of the computer time that this method uses. This method would not be presented except that the author intends calculating the eigenvalues for the nonaxisymmetric stationary perturbation of Poiseuille flow in a circular pipe and the velocity pressure form of the governing equations will be used in that case. It is therefore helpful to check that this form gives the correct answer in the plane case.

The results obtained using the above formulation agree with those in the previous section but an extra real eigenvalue is calculated for the case U, P odd, V even. The eigenvalues are given in Table I for Reynolds numbers $R = 0.25$–10.0. The eigenvalue value is negative so will be associated with upstream disturbances. It will be noticed that $\alpha/R \approx -0.75$. An eigenvalue behaving like $-0.75/R$ could exist for R greater than 10 but here are several real negative eigenvalues whose modulus is less than this one. This extra real eigenvalue is particularly interesting when R is less than about 6 because it is then the only real eigenvalue. The existance of this eigenvalue was checked by increasing the number of Chebyshev polynomials and it is concluded that

TABLE I

Reynolds Number R	Eigenvalue α
0.25	−0.1874
0.5	−0.3748
1.0	−0.7495
2.5	−1.8734
5.0	−3.7438
10.0	−7.4636

these eigenvalues exist in this formulation but not in the formulation of Section 2 or that of [1].

4. Concluding Remarks

This paper shows that the Chebyshev series spectral method can be used to calculate all the eigenvalues (both complex and real) of the stationary perturbation of Poiseuille flow. The method described in Section 2 uses the least computer time. It is not possible to given meaningful computation times because this work of [1] was started on a Cyber 73, continued on an ICL 2980 and the work in this note computed on an ICL 2988.

References

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